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A RANK THEOREM FOR ANALYTIC MAPS BETWEEN POWER SERIES SPACES

by Herwig HAUSER *and* Gerd MÜLLER

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Introduction

Purpose of this paper is to give a criterion for linearizing analytic mappings $f: \mathbf{C}\{x\}^p \to \mathbf{C}\{x\}^q$ between spaces of convergent power series in n variables. Such mappings often occur as defining equations of specific subsets of power series spaces relevant to singularity theory and local analytic geometry. One wants to show that these subsets actually are submanifolds. To this end the defining map has to be linearized locally by analytic automorphisms of source and target.

For maps between finite-dimensional or Banach spaces the appropriate tool is the Rank Theorem: maps of constant rank have smooth fibers. It is established as a direct consequence of the Inverse Mapping Theorem. But this is known to fail for spaces more general than Banach spaces.

The same happens in our situation: we can indicate simple examples of analytic maps between power series spaces which are not local analytic isomorphisms at a given point although the tangent map at this point is an isomorphism. Extra assumptions will be necessary. There are, in fact, extensions of the Inverse Mapping Theorem beyond the frame of Banach spaces, see [C, Ham, L, P3]. They involve technical conditions which it seems hard to verify in our situation.

Instead, we shall develop an Inverse Mapping Theorem which is adapted to the very concrete context we are working in. It allows, using a detailed version of the Division Theorem for modules of power series, to reach our main objective, the Rank Theorem. Let $E = \mathcal{O}_n^p$, $F = \mathcal{O}_n^q$ and $E_c = \{ a \in E, \operatorname{ord}_0 a \geqslant c \}$ for $c \in \mathbf{N}$, where $\mathcal{O}_n$ denotes the ring of convergent power series in n variables with complex coefficients.

Rank Theorem. — Let $g(x,y) \in \mathcal{O}_{n+p}^q$ be a vector of convergent power series in two sets of variables x and y satisfying $g(x, 0) = 0$. There is a $c_0 \in \mathbf{N}$ such that for any $c \geqslant c_0$ for which the induced map

$$f: E_c \to F : a \mapsto g(x, a(x))$$

has constant rank at 0 there are local analytic isomorphisms u of E_c at 0 and v of F at 0 linearizing f, i.e. such that

$$vfu^{-1} = T_0 f.$$

The number c_0 is related to the highest order occurring in a standard basis of the image of the tangent map $T_0 f$. Constant rank is equivalent to saying that the kernels of the tangent maps $T_a f$ form a flat family of modules. The linearizing automorphisms will in general not be given by substitution.

The above Rank Theorem allows to apply methods of differential calculus in the infinite-dimensional context. For instance, we shall prove that for the infinite-dimensional Lie group $G = \operatorname{Aut}(\mathbf{C}^n, 0)$ of local analytic automorphisms of $(\mathbf{C}^n, 0)$ the stabilizer group G_h of a given power series h under the natural action of G on $\mathcal{O}_n$ is a submanifold and thus a Lie subgroup (Theorem 4.1). Its Lie algebra consists of the vector fields which annihilate h. Similar results hold for the contact group K and show that hypersurfaces $X \subset (\mathbf{C}^n, 0)$ are determined up to isomorphism by their Lie group of embedded automorphisms (Theorem 4.2).

Example. — For $h(x,y) = x.y$ one has to investigate the equation $y.a + x.b + a.b = 0$ with unknown series a and b. The corresponding map

$$f: (x,y).\mathbf{C}\{x,y\}^2 \to \mathbf{C}\{x,y\} : (a, b) \mapsto y.a + x.b + a.b$$

is of constant rank at 0. The tangent maps of f are given by

$$T_{(a,b)} f(v, w) = v.(y + b) + w.(x + a).$$

Their kernels form a flat family. The Rank Theorem applies and f can be linearized at 0 into $(a, b) \mapsto y.a + x.b$.

Let us briefly indicate how the Rank Theorem above is proven. As in all theorems of this type one estimates the size of the terms of order $\geqslant 2$ in comparison to the tangent map at 0. In the present situation, the spaces are filtered by Banach spaces. The map f is shown to respect these filtrations and thus induces by restriction Banach analytic maps. Each of those can be linearized locally by the Rank Theorem for Banach spaces.

The point is to show that the linearizing automorphisms of each level glue together to well-defined analytic automorphisms u and v of E_c and F at 0. It is here that the restriction of f to $E_c \subset E$ is used. Moreover, in the construction of u and v, scissions of $\mathcal{O}_n$-linear maps $E \to F$ are needed and have to be estimated. This relies on the Division Theorem for power series in the version of Grauert and Hironaka with norm estimates.

The first three sections present the notion of constant rank and the main results. This is applied in section 4 to show the smoothness of stabilizer groups. Sections 5 to 7 are auxiliary and collect technical tools needed in the proofs.

The results of the present paper were announced in [HM1]. The first named author thanks the members of the Max-Planck-Institut in Bonn for their hospitality during part of the work on this article.

1. Constant rank and flatness

For our purposes a definition of constant rank weaker than the one given in Bourbaki [B] is convenient. For an analytic $f: U \to F$ with $U \subset E$ open we do not assume that there is some closed J such that

$$\operatorname{Im} T_a f \oplus J = F$$

holds pointwise for all a near a_0. We only require that this equality holds analytically in a, namely that any analytic curve in F decomposes uniquely as a sum of curves in $\operatorname{Im} T_a f$ and J. Let us give this sentence a precise meaning. For a topological vector space F (Hausdorff, locally convex and sequentially complete), consider the tangent bundle $TF = F \times F$ and subbundles $J_F = F \times J$ where $J \subset F$ is a closed subspace, together with the tangent bundle map

$$Tf: TE|_U \to TF : (a, b) \mapsto (f(a), T_a f(b)).$$

For the germ of an analytic curve $\gamma : (\mathbf{C}, 0) \to E$ let $\Gamma_\gamma(TE)$ denote the vector space of germs of analytic sections of TE over γ, i.e. germs of analytic maps $(\mathbf{C}, 0) \to TE : t \mapsto (\gamma(t), b(t))$, with induced map

$$Tf: \Gamma_\gamma(TE) \to \Gamma_{f\gamma}(TF) : (\gamma, b) \mapsto (f\gamma, (T_\gamma f)\, b).$$

Define $\Gamma_{f\gamma}(J_F)$ in an obvious way as space of sections over $f\gamma$ with values in J. Then f is said to have *constant rank* at $a_0 \in U$ if the image of $T_{a_0} f$ has a topological direct complement J

$$\operatorname{Im} T_{a_0} f \oplus J = F$$

such that for all analytic germs $\gamma : (\mathbf{C}, 0) \to E$ with $\gamma(0) = a_0$

$$Tf(\Gamma_\gamma(TE)) \oplus \Gamma_{f\gamma}(J_F) = \Gamma_{f\gamma}(TF)$$

(as an algebraic direct sum). This means that any analytic section σ of TF over $f\gamma$ can be written uniquely as a sum of a section τ of J_F over $f\gamma$ and a composition $(Tf)\,\rho$ where ρ is a section of TE over γ

$$\sigma = (Tf)\,\rho + \tau.$$

We sometimes say that f has constant rank at a_0 w.r.t. J. On the other hand, f is called *flat* at a_0 if for all analytic germs $\gamma : (\mathbf{C}, 0) \to \mathrm{E}$ with $\gamma(0) = a_0$ the evaluation map

$$\mathrm{Ker}(Tf : \Gamma_\gamma(T\mathrm{E}) \to \Gamma_{f\gamma}(T\mathrm{F})) \to \mathrm{Ker}(T_{a_0}f : \mathrm{E} \to \mathrm{F})$$

is surjective. This means that for any $b \in \mathrm{E}$ with $\mathrm{T}_{a_0}f(b) = 0$ there is an analytic germ $t \mapsto b_t$ in E with $b_0 = b$ and $T_{\gamma(t)}f(b_t) = 0$ for all t (" lifting of relations "). In the special case where the tangent map $T_{a_0}f$ of f at a_0 is injective, f is automatically flat at a_0.

Proposition **1.1.** — *Let* $f : \mathrm{U} \to \mathrm{F}$, $\mathrm{U} \subset \mathrm{E}$ *open, be analytic.*

a) If f *has constant rank at* $a_0 \in \mathrm{U}$ *it is flat at* a_0.

b) Assume that f *is flat at* $a_0 \in \mathrm{U}$ *and that the image of* $T_{a_0}f$ *admits a topological direct complement* J *in* F. *Then for every analytic germ* $\gamma : (\mathbf{C}, 0) \to \mathrm{E}$ *with* $\gamma(0) = a_0$ *one has*

$$Tf(\Gamma_\gamma(T\mathrm{E})) \cap \Gamma_{f\gamma}(\mathrm{J_F}) = 0.$$

Proof. — *a)* Let $\gamma : (\mathbf{C}, 0) \to \mathrm{E}$ with $\gamma(0) = a_0$ be analytic, and fix $b \in \mathrm{Ker}\, T_{a_0}f$. The analytic germ $t \mapsto T_{\gamma(t)}f(b)$ can be written

$$T_{\gamma(t)}f(b) = t.\widetilde{b}(t)$$

with an analytic germ $\widetilde{b} : (\mathbf{C}, 0) \to \mathrm{F}$ using power series expansion and $T_{a_0}f(b) = 0$. Then $t \mapsto (f(\gamma(t)), \widetilde{b}(t))$ is contained in $\Gamma_{f\gamma}(T\mathrm{F})$. Suppose that f has constant rank at a_0 w.r.t. J. The existence part of the constant rank condition allows to write

$$\widetilde{b}(t) = T_{\gamma(t)}f(c_t) + d_t$$

with analytic germs $t \mapsto c_t$ in E and $t \mapsto d_t$ in J. Then the analytic germ $t \mapsto b_t = b - t.c_t$ in E satisfies $b_0 = b$ and $T_{\gamma(t)}f(b_t) = t.d_t \in \mathrm{J}$ for all t. The uniqueness part of the constant rank condition implies $T_{\gamma(t)}f(b_t) = 0$ for all t.

b) Let b_t be an analytic curve in E with $c_t = T_{\gamma(t)}f(b_t) \in \mathrm{J}$ for all t. In particular, $c_0 = T_{a_0}f(b_0) = 0$. By assumption there is an analytic curve $\widetilde{b}_t$ in E with $\widetilde{b}_0 = b_0$ and $T_{\gamma(t)}f(\widetilde{b}_t) = 0$ for all t. This yields

$$T_{\gamma(t)}f(b_t - \widetilde{b}_t) = c_t.$$

Since $b_0 - \widetilde{b}_0 = 0$ one can factor out t on both sides and then prove by induction on the order that $c_t = 0$ for all t. This proves the Proposition.

In the cases we shall be concerned with the existence part of the constant rank condition will be automatic (and thus, roughly speaking, constant rank will be equivalent to flatness).

Let $l : E \to F$ be a continuous linear map. A *scission* of l is a continuous linear map $\sigma : F \to E$ with $l\sigma l = l$. Then $\mathrm{id} - \sigma l : E \to E$ and $l\sigma : F \to F$ are continuous projectors onto $\operatorname{Ker} \sigma l = \operatorname{Ker} l$ and $\operatorname{Im} l\sigma = \operatorname{Im} l$ inducing decompositions $E = \operatorname{Ker} l \oplus \operatorname{Im}(\mathrm{id} - \sigma l)$ and $F = \operatorname{Im} l \oplus \operatorname{Ker} l\sigma$.

If E and F are Banach scales (cf. sec. 6) l admits a scission if and only if $\operatorname{Im} l$ and $\operatorname{Ker} l$ admit topological direct complements in F, respectively E. If $\operatorname{Im} l \oplus J = F$ and $\operatorname{Ker} l \oplus L = E$ then $l|_L : L \to \operatorname{Im} l$ is bijective with continuous inverse (the Closed Graph Theorem holds in Banach scales). If $\pi : F \to \operatorname{Im} l$ is the projector with kernel J then $\sigma = (l|_L)^{-1} \pi$ is a scission of l.

Proposition **1.2.** — *Let* E *and* F *be Banach scales,* $U \subset E$ *be open and* $f : U \to F$ *be an analytic map which is compatible at* $a_0 \in U$. *Assume that* $l = T_{a_0} f$ *admits a compatible scission* $\sigma : F \to E$ *with corresponding decomposition*

$$F = I \oplus J, \qquad I = \operatorname{Im} l, \qquad J = \operatorname{Ker} l\sigma.$$

a) For every analytic curve $\gamma : (\mathbf{C}, 0) \to E$ *with* $\gamma(0) = a_0$ *one has*

$$\Gamma_{f\gamma}(TF) = Tf(\Gamma_\gamma(TE)) + \Gamma_{f\gamma}(J_F).$$

b) If f *is flat at* a_0 *(e.g. if* f *has constant rank at* a_0 *w.r.t. some other complement* J' *of* I*)* f *has constant rank at* a_0 *w.r.t.* J.

Proof. — As *b)* is immediate from *a)* by Proposition 1.1 *b)* we have only to prove part *a)*. Let $b(t)$ be an analytic curve in F. By [P2, Proposition 2] the germs $\gamma : (\mathbf{C}, 0) \to E$ and $b : (\mathbf{C}, 0) \to F$ have values in E_s, respectively F_s for sufficiently small s and are analytic as E_s-, respectively F_s-valued map germs. We are thus reduced to the case where E and F are Banach spaces. Let $L(E, F)$ be the Banach space of continuous linear maps $E \to F$. By [U, 1.8] the map $U \to L(E, F) : b \mapsto T_b f$ is analytic. Moreover, the invertible elements of $L(F, F)$ form an open subset, and the inversion mapping $\varphi \mapsto \varphi^{-1}$ is analytic, see [U, 2.7]. For $a \in U$ consider

$$\varphi_a = T_a f \sigma l \sigma + \mathrm{id}_F - l\sigma \in L(F, F).$$

As $\varphi_{a_0} = \mathrm{id}$ we may assume after shrinking U that all φ_a are invertible. We see that $a \mapsto \varphi_a^{-1}$ is analytic. Then $c(t) = \sigma l \sigma \varphi_{\gamma(t)}^{-1} b(t)$ and $d(t) = (\mathrm{id}_F - l\sigma)\, \varphi_{\gamma(t)}^{-1} b(t)$ define analytic curves in E, respectively J satisfying

$$b(t) = T_{\gamma(t)} f(c(t)) + d(t).$$

This proves the Proposition. In section 5, remark *b)*, we deduce:

Corollary. — *Let* $f : U \to \mathcal{O}_n^q$ *be an analytic map given in a neighborhood* $U \subset \mathcal{O}_n^p$ *of* 0 *by substitution in a power series* $g(x, y)$. *Then* f *has constant rank at* 0 *if and only if* f *is flat at* 0.

Examples. — *a)* $f : \mathcal{O}_1 \to \mathcal{O}_1 : a \mapsto xa + a^2$ has Gâteaux-differentials

$$T_a f : \mathcal{O}_1 \to \mathcal{O}_1 : b \mapsto (x + 2a)\, b.$$

Since $T_0 f$ is injective, f is flat at 0, hence of constant rank. But $\operatorname{Im} T_a f = \mathcal{O}_1$ for $a(0) \neq 0$ and $\operatorname{Im} T_0 f = (x)$. Thus for a close to 0 the images of the tangent maps $T_a f$ do not have a simultaneous direct complement.

b) $f: \mathcal{O}_1 \to \mathcal{O}_1 : a \mapsto a + (1/x)(a^2 - a^2(0))$ has Gâteaux-differential $T_0 f = \mathrm{id}$. Again f has constant rank at 0 and for a close to 0 the kernels of the tangent maps $T_a f$ do not have a simultaneous direct complement, see the example of section 6.

2. The Rank Theorem in Banach scales

For the notion of Banach scales and compatible maps see section 6.

Rank Theorem **2.1.** — *Let* $\mathrm{E} = \bigcup_s \mathrm{E}_s$, $\mathrm{F} = \bigcup_s \mathrm{F}_s$ *be Banach scales and* $f: \mathrm{U} \to \mathrm{F}$ *be an analytic map in a neighborhood* $\mathrm{U} \subset \mathrm{E}$ *of* 0 *with* $f(0) = 0$. *Set* $l = T_0 f$, $\mathrm{I} = \operatorname{Im} l$, $f = l + h$. *Assume that* l *admits a scission* $\sigma: \mathrm{F} \to \mathrm{E}$, $l\sigma l = l$, *with the following properties*:

(i) f *has constant rank at* 0 *w.r.t. to the topological direct complement* $\mathrm{J} = \operatorname{Ker} l\sigma$ *of* I.

(ii) $\sigma h: \mathrm{U} \to \mathrm{E}$ *is compatible at* 0 *and its restrictions satisfy*

$$|(\sigma h)_s|_r \leqslant c.r$$

for some constants $0 < c < 1/e(e+1)$ *and* $0 < r$, *and all small* $s > 0$.

Then there are local analytic isomorphisms u *of* E *at* 0 *and* v *of* F *at* 0 *linearizing* f, *i.e. such that*

$$vfu^{-1} = l.$$

If f and σ are compatible with the filtrations, assumption (i) can be replaced by the condition that f has constant rank at 0 (w.r.t. some other complement J′), see Proposition 1.2. If the image of l admits a topological direct complement J then constant rank is necessary for f to be linearizable near 0. In fact, l has constant rank at 0 w.r.t. J since $T_a l = l$ for all $a \in \mathrm{E}$. Observe that (i) plus (ii) imply in particular Bourbaki's rank condition.

Proof. — Set $u = \mathrm{id}_\mathrm{E} + \sigma h : \mathrm{U} \to \mathrm{E}$. Then u is analytic and compatible at 0 with $u(0) = 0$ and $T_0 u = \mathrm{id}_\mathrm{E}$. By the norm estimate in (ii) the Inverse Mapping Theorem 6.2 applies: u is a local analytic automorphism of E at 0. Note that $l\sigma f = l\sigma l + l\sigma h = l + l\sigma h = lu$, hence $l\sigma f u^{-1} = l$ near 0. Replacing f by fu^{-1} we may assume that $l\sigma f = l$. The map $v = \mathrm{id}_\mathrm{F} - (\mathrm{id}_\mathrm{F} - l\sigma) f\sigma l\sigma$ is a local analytic automorphism of F at 0 with inverse $\mathrm{id}_\mathrm{F} + (\mathrm{id}_\mathrm{F} - l\sigma) f\sigma l\sigma$. Assume for the moment that $f = f\sigma l$ near 0. Then

$$\begin{aligned} vf &= f - (\mathrm{id}_\mathrm{F} - l\sigma) f\sigma l\sigma f \\ &= f - (\mathrm{id}_\mathrm{F} - l\sigma) f\sigma l \\ &= f - (\mathrm{id}_\mathrm{F} - l\sigma) f \\ &= l\sigma f = l \end{aligned}$$

near 0 as required. The equality $f = f\sigma l$ follows from the constant rank condition: σ being a scission of l, σl is a linear projection onto a topological direct complement of $K = \operatorname{Ker} l$ in E. It suffices to show that $T_a f|_K = 0$ for all $a \in E$ close to 0. But $l\sigma f = l$ from above implies $f = l + (\mathrm{id}_F - l\sigma) f$, hence $T_a f = l + (\mathrm{id}_F - l\sigma)\, T_a f$. Fix a point a in an absolutely convex open neighborhood of 0 in U, and consider the analytic curve $\gamma : (\mathbf{C}, 0) \to E$, $\gamma(t) = t.a$. For $b \in K = \operatorname{Ker} l$ we get

$$T_{\gamma(t)} f(b) = (\mathrm{id}_F - l\sigma)\; T_{\gamma(t)} f(b),$$

so that

$$(f\gamma, T_\gamma f(b)) \in Tf(\Gamma_\gamma(TE)) \cap \Gamma_{f\gamma}(J_F).$$

By assumption (i) $T_{\gamma(t)} f(b)$ must be 0 for t close to 0. Then $T_a f(b) = 0$ by analytic continuation. This proves $T_a f|_K = 0$ and the Theorem.

Corollary. — Assume that f is compatible at 0 *and that l admits a compatible scission $\sigma : F \to E$ such that*

$$|(\sigma h)_s|_r \leqslant c.r$$

for some constants $0 < c < 1/e(e+1)$ and $0 < r$, and all small $s > 0$.

a) If l is injective f admits locally at 0 *an analytic left inverse g. In particular, f is locally injective.*

b) If l is surjective f admits locally at 0 *an analytic right inverse g. In particular, f is open at* 0.

Proof. — *a)* Since l is injective f is flat at 0 and thus has constant rank at 0. Therefore f can be linearized locally. From the equality $l\sigma l = l$ we obtain an analytic map g with $fgf = f$. Moreover, injectivity of l implies that f is locally injective. Thus $gf = \mathrm{id}_E$.

b) In case l is surjective one has $l\sigma = \mathrm{id}_F$. The beginning of the proof of the Rank Theorem shows that $l\sigma f u^{-1} = l$, hence $fu^{-1}\sigma = l\sigma = \mathrm{id}_F$.

3. The Rank Theorem in power series spaces

Fix a weight vector $\lambda = (\lambda_0, \lambda') \in \mathbf{R}^{1+n}$ with $\mathbf{Z}$-linearly independent components $\lambda_i \geqslant 1$. It induces a Banach scale structure on $\mathcal{O}_n = \bigcup \mathcal{O}_n(s)$ where $\mathcal{O}_n(s)$ are the subspaces on which

$$|b|_s = \Sigma_\alpha |b_\alpha| s^{\lambda'\alpha} \quad \text{for } b = \Sigma_\alpha b_\alpha x^\alpha \in \mathcal{O}_n$$

is finite. Moreover, $\mathcal{O}_n^q$ becomes a Banach scale by

$$|b|_s = \Sigma_i s^{\lambda_0(i-1)} |b_i|_s \quad \text{for } b = (b_1, \ldots, b_q) \in \mathcal{O}_n^q.$$

For subsets $M \subset \mathcal{O}_n^q$ we set $M_s = M \cap (\mathcal{O}_n^q)_s$. For a submodule I of $\mathcal{O}_n^q$ denote by $\beta(I)$ the maximal weighted order of a minimal standard basis of I, cf. section 5. For $\alpha \in \mathbf{N}^n$ set $|\alpha| = \alpha_1 + \ldots + \alpha_n$. Let m_n denote the maximal ideal of $\mathcal{O}_n$.

Rank Theorem **3.1.** — *Let* $E = \mathcal{O}_n^p$, $E_c = m_n^c . \mathcal{O}_n^p$ *and* $F = \mathcal{O}_n^q$. *Let* $g(x,y) \in \mathcal{O}_{n+p}^q$ *be a vector of convergent power series in two sets of variables* x *and* y *with* $g(x, 0) \equiv 0$. *Let* $I = (\partial_y g(x, 0)) \subset \mathcal{O}_n^q$ *be the submodule generated by the partial derivatives w.r.t.* y, *and write* $g(x,y) = \Sigma\, m_\alpha(x)\, y^\alpha$. *For any* $c \in \mathbf{N}$ *with*

$$\beta(m_n^c\, I) \leqslant \operatorname{ord}(m_\alpha) + c . | \alpha |$$

for all $| \alpha | \geqslant 2$ *and such that*

$$f : E_c \to F : a \mapsto g(x, a(x))$$

has constant rank at 0 *there are local analytic isomorphisms* u *and* v *of* E_c *and* F *at* 0 *linearizing* f

$$vfu^{-1} = T_0 f.$$

Remark. — There always exists a c_0 with $\beta(m_n^c\, I) \leqslant \operatorname{ord}(m_\alpha) + c . | \alpha |$ for $c \geqslant c_0$, see Remark *c)*, section 5.

Proof. — We reduce to the Rank Theorem for Banach scales. Equip $E_c = m_n^c . \mathcal{O}_n^p$ with

$$|| a ||_s = s^{-c}\, \Sigma_i \, | a_i |_s \quad \text{for } a = (a_1, \ldots, a_p) \in E_c.$$

The pseudonorm is contractive, $|| a ||_s \leqslant || a ||_{s'}$ for $a \in E_c$ and $s < s'$, because $\lambda_i \geqslant 1$. Set $f = l + h$ with $l = T_0 f$ and $h : E_c \to F$ of order $\geqslant 2$. Let $h_s = \Sigma\, h_{sk}$ be the power series expansion of h_s in continuous homogeneous polynomials and set $| h_s |_r = \Sigma \, | h_{sk} | \, r^k$ for $r > 0$, cf. [U]. We prove:

a) $l : E_c \to F$ admits a compatible scission $\sigma : F \to E_c$ satisfying $| \sigma_s | \leqslant c_1 . s^{-\beta}$ for some constant $c_1 > 0$ and all small $s > 0$ where $\beta = \beta(m_n^c\, I)$.

b) For any constant $c_2 > 0$ there is an $r > 0$ such that, for all small $s > 0$,

$$| h_s |_r \leqslant c_2 . s^\beta . r.$$

c) There are constants $0 < c_3 < 1/e(e + 1)$ and $r > 0$ such that, for all small $s > 0$,

$$| (\sigma h)_s |_r \leqslant c_3 . r.$$

Let us show *a)*: Let N be the number of monic monomials of degree c in the variables $x_1, \ldots, x_n$. Let $\rho : \mathcal{O}_n^{N . p} \to E_c = m_n^c . \mathcal{O}_n^p$ be the canonical surjective $\mathcal{O}_n$-linear map. Provide $\mathcal{O}_n^{N . p}$ with the Banach scale structure given by

$$| a |_s = \Sigma_i \, | a_i |_s, \quad \text{for } a = (a_1, \ldots, a_{N . p}) \in \mathcal{O}_n^{N . p}.$$

Then ρ is compatible and satisfies $| \rho_s | \leqslant 1$ for all small $s > 0$, since $\lambda_i \geqslant 1$ and thus $| x^\alpha |_s \leqslant s^c$ for monic monomials x^α of degree c. By Theorem 5.2 the $\mathcal{O}_n$-linear map

$$l\rho : \mathcal{O}_n^{N . p} \to F = \mathcal{O}_n^q$$

admits a compatible scission $\tau : \mathcal{O}_n^q \to \mathcal{O}_n^{N . p}$ with

$$| \tau_s | \leqslant c_1 . s^{-\beta}$$

for some constant $c_1 > 0$ and all small $s > 0$. Then $\sigma := \rho\tau : \mathrm{F} \to \mathrm{E}_c$ is a compatible scission of l with

$$| \sigma_s | \leqslant c_1 . s^{-\beta},$$

proving *a)*. To see *b)*, note first that one has, by definition of $| h_s |_r$,

$$\begin{aligned} | h_s |_r &= \Sigma_{k \geqslant 2} \sup_{\|a\|_s \leqslant 1} | \Sigma_{|\alpha| = k}\, m_\alpha\, a^\alpha |_s . r^k \\ &\leqslant \Sigma_{k \geqslant 2} \sup_{\Sigma |a_i|_s \leqslant s^c} \Sigma_{|\alpha| = k} | m_\alpha |_s \, | a^\alpha |_s . r^k \leqslant \Sigma_{|\alpha| \geqslant 2} | m_\alpha |_s . s^{c|\alpha|} . r^{|\alpha|}. \end{aligned}$$

Now set $d_\alpha(s) = | m_\alpha |_s\, s^{c|\alpha|}\, s^{-\beta}$. It follows from $\lambda_i \geqslant 1$ and the assumption on β that $d_\alpha(s)$ decreases as s goes to 0. Moreover

$$\Sigma_{|\alpha| \geqslant 2}\, d_\alpha(s_0) . r^{|\alpha|} \leqslant s_0^{-\beta} . \Sigma_{|\alpha| \geqslant 2} | m_\alpha |_{s_0} . r^{|\alpha|}$$

converges for small s_0 and r since $g(x,y)$ converges. Therefore for any $c_2 > 0$ there is an $r > 0$ such that

$$| h_s |_r \leqslant s^\beta . \Sigma_{|\alpha| \geqslant 2}\, d_\alpha(s_0) . r^{|\alpha|} \leqslant s^\beta . c_2 . r$$

for all $s \leqslant s_0$. This is *b)*, and *c)* follows from *a)* and *b)*. Thus 2.1 applies and the Theorem is proven.

Examples. — *a)* Consider $g(x,y) = xy + y^2$ where $\mathrm{I}_c = (x)^{c+1}$. As

$$f : (x)^c \to \mathcal{O}_1 : a \mapsto xa + a^2$$

has constant rank at 0 it can be linearized locally near 0 for $c \geqslant 1$.

b) The map $f : \mathcal{O}_2 \to \mathcal{O}_2 : a \mapsto xya + (x^2 + y^2)\, a^2$ has constant rank at 0. Furthermore, $\mathrm{I} = (xy)$, and $\beta(\mathrm{I}) > 2$. Hence Theorem 3.1 cannot be applied to linearize f on $\mathcal{O}_2$. But one can use the Rank Theorem for Banach scales directly. Provide $\mathcal{O}_2$ with the Banach scale structure associated to $\lambda = (\lambda_1, \lambda_2) = (1, 1)$. These weights are not permitted in 3.1. The natural projection $\mathcal{O}_2 \to \mathrm{I}$ and division by xy induce a compatible scission σ. One calculates

$$| \sigma_s | \leqslant s^{-2} \quad \text{and} \quad | h_s |_r \leqslant 2 s^2 r^2$$

for the quadratic part $h(a) = (x^2 + y^2)\, a^2$. Thus the norm estimate (ii) of Theorem 2.1 is satisfied and $f : \mathcal{O}_2 \to \mathcal{O}_2$ can be linearized locally at 0.

4. Application: stabilizer groups

We apply the preceding results to prove the smoothness of stabilizer groups. Let $\mathrm{G} = \mathrm{Aut}(\mathbf{C}^n, 0)$ be the group of germs of analytic automorphisms of $(\mathbf{C}^n, 0)$ and p be a natural number. The contact group K is the semidirect product

$$\mathrm{K} = \mathrm{Aut}(\mathbf{C}^n, 0) \ltimes \mathrm{GL}_p(\mathcal{O}_n)$$

defined through the right action of G on $\mathrm{GL}_p(\mathcal{O}_n)$. It is an open subset of $\mathbf{k} = m_n \mathcal{O}_n^n \oplus \mathcal{O}_n^{p^2}$. By Proposition 6.1, one can prove that K and G are infinite-dimensional Lie groups in the sense of [Mi, sec. 5]. The tangent space $\mathbf{k}$ of K has a natural Lie algebra structure $[v, w] = \mathrm{ad}(v)\, w$ where $\mathbf{k} \to \mathbf{k} : v \mapsto \mathrm{ad}(v)\, w$ is the tangent map of $\mathrm{K} \to \mathbf{k} : g \mapsto \mathrm{Ad}(g)\, w$ and $\mathrm{Ad}(g) : \mathbf{k} \to \mathbf{k}$ is the tangent map of conjugation with g. Identifying elements of $\mathcal{O}_n^n$ with derivations of $\mathcal{O}_n$ the Lie algebra of G is $\mathbf{g} = \{ \mathrm{D} \in \mathrm{Der}\, \mathcal{O}_n, \mathrm{D}(m_n) \subset m_n \}$ with the usual bracket $[\mathrm{D}, \mathrm{E}] = \mathrm{DE} - \mathrm{ED}$. The bracket on $\mathbf{k}$ equals

$$[(\mathrm{D}, v), (\mathrm{E}, w)] = ([\mathrm{D}, \mathrm{E}], [v, w] + \mathrm{D}w - \mathrm{E}v).$$

The contact group acts analytically on the space $\mathcal{O}_n^p$ of rows from the right by $h(\varphi, a) = (h \circ \varphi)\,.\,a$.

Theorem **4.1.** — *For $h \in \mathcal{O}_n^p$ the stabilizer group*

$$\mathrm{K}_h = \{ (\varphi, a) \in \mathrm{K}, (h \circ \varphi)\,.\,a = h \}$$

is a submanifold of K, *hence a Lie subgroup. Its Lie algebra is*

$$\mathbf{k}_h = \{ (\mathrm{D}, v) \in \mathbf{k}, \mathrm{D}h + hv = 0 \}.$$

Proof. — *a)* For fixed $c \in \mathbf{N}$ consider the finite-dimensional algebra $\mathrm{V}_c = \mathcal{O}_n / m_n^c$ and the finite-dimensional algebraic group $\mathrm{A}_c = \mathrm{Aut}\, \mathrm{V}_c \ltimes \mathrm{GL}_p(\mathrm{V}_c)$. There is a natural surjective homomorphism of Lie groups $\pi_c : \mathrm{K} \to \mathrm{A}_c$. Let $\mathrm{K}_c = \mathrm{Ker}\, \pi_c$, $\mathrm{H} = \mathrm{K}_h$ and $\mathrm{H}_c = \mathrm{H} \cap \mathrm{K}_c$. We apply the Rank Theorem to show that for c sufficiently large, H_c is a submanifold of K_c. Writing $\varphi(x) = x + \psi(x)$, $a(x) = 1 + b(x)$ we identify K_c as a manifold with the space $\mathrm{E}_c = m_n^c\,.\,\mathcal{O}_n^{n+p^2}$. Then H_c equals the zero-fiber of the analytic map

$$f : \mathrm{E}_c \to \mathcal{O}_n^p, f(\psi, b) = h(x + \psi)\,.\,(1 + b) - h.$$

Its Gâteaux-differential is given by

$$T_{(\psi, b)} f(\mathrm{D}, v) = \mathrm{D}h(x + \psi)\,.\,(1 + b) + h(x + \psi)\,.\,v.$$

If $T_0 f(\mathrm{D}, v) = 0$ then

$$T_{(\psi, b)} f(\mathrm{D}(x + \psi), v(x + \psi)\,.\,(1 + b)) = 0.$$

Hence f is flat at 0. The Rank Theorem implies that f can be linearized locally.

b) Similarly as in [Mül, sec. 2] it is shown using Artin's Approximation Theorem [A] that the image $\mathrm{B}_c = \pi_c(\mathrm{H})$ is an algebraic subgroup of A_c. Below we shall see that $\pi_c : \mathrm{K} \to \mathrm{A}_c$ admits locally at $1 \in \mathrm{A}_c$ an analytic section $\sigma : \mathrm{U} \to \mathrm{K}$ which restricts to a section $\mathrm{U} \cap \mathrm{B}_c \to \mathrm{H}$ of the restriction $\mathrm{H} \to \mathrm{B}_c$. Then

$$\pi_c^{-1}(\mathrm{U}) \to \mathrm{U} \times \mathrm{K}_c : g \mapsto (\pi_c(g), (\sigma\pi_c(g))^{-1}\, g)$$

is an analytic isomorphism with inverse

$$\mathrm{U} \times \mathrm{K}_c \to \pi_c^{-1}(\mathrm{U}) : (g_c, g) \mapsto \sigma(g_c)\, g.$$

It restricts to a bijective map

$$\pi_c^{-1}(\mathrm{U}) \cap \mathrm{H} \to (\mathrm{U} \cap \mathrm{B}_c) \times \mathrm{H}_c.$$

Therefore H is a submanifold of K.

c) To prove the existence of σ let $\mathbf{a}_c$ be the Lie algebra of A_c and $\dot{\pi}_c : \mathbf{k} \to \mathbf{a}_c$ the tangent map of π_c. Moreover let $\mathbf{h} = \mathbf{k}_h$ and $\mathbf{b}_c = \dot{\pi}_c(\mathbf{h})$. Using again the Approximation Theorem it is checked that $\mathbf{b}_c$ is the Lie algebra of B_c. By solving an appropriate initial value problem one can show that the Lie group K admits an exponential map $\exp : \mathbf{k} \to \mathrm{K}$, i.e., an analytic map such that for all $(\mathrm{D}, v) \in \mathbf{k}$ the map

$$\mathbf{C} \to \mathrm{K} : t \mapsto \exp t(\mathrm{D}, v)$$

is a one-parameter subgroup with initial velocity vector (D, v), see also [P1, sec. 1]. Since the exponential map of the finite-dimensional Lie group A_c is a local analytic isomorphism near $0 \in \mathbf{a}_c$ and since the exponential map of K maps $\mathbf{h}$ into H, see step *d)* below, the existence of the section σ follows from the corresponding infinitesimal assertion: $\dot{\pi}_c : \mathbf{k} \to \mathbf{a}_c$ admits a continuous linear section $\tau : \mathbf{a}_c \to \mathbf{k}$ which restricts to a section $\mathbf{b}_c \to \mathbf{h}$ of the restriction $\mathbf{h} \to \mathbf{b}_c$. For this take the canonical section $\tau_0 : \mathbf{a}_c \to \mathbf{k}$ which maps $(\mathrm{D}_c, v_c) \in \mathbf{a}_c$ onto the unique polynomial of degree $\leqslant c - 1$ in the fiber of (D_c, v_c). Then adjust τ_0 by defining

$$\tau = (\mathrm{id}_{\mathbf{k}} - \rho\varepsilon) \circ \tau_0 : \mathbf{a}_c \to \mathbf{k}$$

where $\varepsilon : \mathbf{k} \to \mathcal{O}_n^p$ is given by $\varepsilon(\mathrm{D}, v) = \mathrm{D}h + hv$ and $\rho : \mathcal{O}_n^p \to \mathbf{k}_c$ is a scission of ε restricted to $\mathbf{k}_c = \operatorname{Ker} \dot{\pi}_c$. This τ is a continuous linear section of $\dot{\pi}_c$. To show that it maps $\mathbf{b}_c$ into $\mathbf{h}$, take $(\mathrm{D}_c, v_c) \in \mathbf{b}_c$ and let $(\mathrm{D}, v) = \tau_0(\mathrm{D}_c, v_c) + (\mathrm{D}', v')$ be an element of $\mathbf{h}$ mapped onto (D_c, v_c). Then $(\mathrm{D}', v') \in \mathbf{k}_c$ and $\varepsilon\tau_0(\mathrm{D}_c, v_c) = -\varepsilon(\mathrm{D}', v')$. This implies $\varepsilon\tau(\mathrm{D}_c, v_c) = 0$ and $\tau(\mathrm{D}_c, v_c) \in \mathbf{h}$.

d) It remains to show that the Lie algebra of K_h is $\mathbf{k}_h$ and that exp maps $\mathbf{k}_h$ into K_h. Clearly

$$\{(\mathrm{D}, v) \in \mathbf{k}, \exp t(\mathrm{D}, v) \in \mathrm{K}_h \text{ for all } t\} \subset T_1 \mathrm{K}_h \subset \mathbf{k}_h.$$

Conversely, if $(\mathrm{D}, v) \in \mathbf{k}_h$, calculation shows that

$$\frac{d}{dt}(h \circ \varphi_t) . a_t = 0 \quad \text{for } (\varphi_t, a_t) = \exp t(\mathrm{D}, v).$$

Therefore $\exp t(\mathrm{D}, v) \in \mathrm{K}_h$. This proves the Theorem.

Let now $\mathrm{X} \subset (\mathbf{C}^n, 0)$ be a reduced hypersurface defined as zero-fiber $\mathrm{X} = h^{-1}(0)$ of some $h \in \mathcal{O}_n$. Let

$$\mathrm{G}_\mathrm{X} = \{ \varphi \in \mathrm{G}, \varphi(\mathrm{X}) = \mathrm{X} \}$$

be the group of embedded automorphisms of X. Thus

$$\mathrm{G}_\mathrm{X} = \{ \varphi \in \mathrm{G}, h \circ \varphi \in \mathcal{O}_n . h \}.$$

For $\varphi \in G_X$ the factor $a \in \mathcal{O}_n$ such that $h \circ \varphi = h.a$ is a unit which is uniquely determined by φ. Therefore the projection $K \to G : (\varphi, a) \mapsto \varphi$ induces an isomorphism of groups $K_h \xrightarrow{\sim} G_X$. Thus we may view G_X as a Lie group. If X is isomorphic to another hypersurface $Y \subset (\mathbf{C}^n, 0)$, say defined by $g \in \mathcal{O}_n$, then h and g are in the same K-orbit: $g = (h \circ \varphi).a$ for some $(\varphi, a) \in K$. Conjugation with (φ, a) induces an isomorphism $K_h \xrightarrow{\sim} K_g$ of Lie groups. In particular, the Lie group structure on G_X is independent of the chosen equation h, and isomorphic hypersurfaces X, Y have isomorphic Lie groups G_X, G_Y. The converse is also true:

Theorem **4.2.** — *Two reduced hypersurfaces* $X, Y \subset (\mathbf{C}^n, 0)$, $n \geqslant 3$, *are isomorphic if and only if their Lie groups* G_X, G_Y *of embedded automorphisms are isomorphic.*

Proof. — Let $\mathbf{D}_{X,0} = \{ D \in \mathbf{g}, Dh \in h.\mathcal{O}_n \}$. The projection $\mathbf{k} = \mathbf{g} \oplus \mathcal{O}_n \to \mathbf{g}$ induces a continuous isomorphism of Lie algebras $\mathbf{k}_h \xrightarrow{\sim} \mathbf{D}_{X,0}$. Both $\mathbf{k}_h$ and $\mathbf{D}_{X,0}$ are $\mathcal{O}_n$-submodules of $\mathbf{k}$, respectively $\mathbf{g}$, hence closed subspaces. By the Closed Graph Theorem [Gr, chapter 4.1.5, Theorem 2] we conclude that $\mathbf{k}_h \xrightarrow{\sim} \mathbf{D}_{X,0}$ is even an isomorphism of topological vector spaces. Therefore, if $f: G_X \to G_Y$ is an isomorphism of Lie groups, the tangent map of f at the unit induces an isomorphism $\mathbf{D}_{X,0} \xrightarrow{\sim} \mathbf{D}_{Y,0}$ of topological Lie algebras. By [HM2, part II, Theorem and Comments *d), e)* in section 1] the topological Lie algebra $\mathbf{D}_{X,0}$ determines X up to analytic isomorphism. Therefore X and Y are isomorphic.

5. Scissions of $\mathcal{O}_n$-linear maps

In order to construct and control scissions of $\mathcal{O}_n$-linear maps we need a version of the Grauert-Hironaka Division Theorem with special attention paid to norm estimates [Ga], [Hau].

For fixed $\lambda = (\lambda_0, \lambda') \in \mathbf{R}_+^{1+n}$ with $\mathbf{Z}$-linearly independent components λ_i, elements of $\mathbf{N}^{1+n}$ are ordered by $i\alpha < j\beta$ if $\lambda(i\alpha) < \lambda(j\beta)$, where $\lambda(i\alpha) = \lambda_0(i-1) + \Sigma_k \lambda_k \alpha_k$. Define a total order on the set of monic monomial vectors $(0, \ldots, 0, x^\alpha, 0, \ldots, 0)$ of $\mathcal{O}_n^q$ by setting $(0, \ldots, 0, x^\beta, 0, \ldots, 0) < (0, \ldots, 0, x^\alpha, 0, \ldots, 0)$ if $j\beta < i\alpha$ where i and j denote the position of x^α and x^β. Denote by $\operatorname{in}(a)$ and $\operatorname{in}(I)$ the initial monomial vector and initial module of elements and submodules of $\mathcal{O}_n^q$. Set

$$\Delta(I) = \{ b \in \mathcal{O}_n^q, \text{ no monomial of } b \text{ belongs to } \operatorname{in}(I) \}.$$

For a standard basis $m_1, \ldots, m_p$ of I with initial terms $\mu_1, \ldots, \mu_p$, partition the support of $\operatorname{in}(I)$ as a disjoint union $\bigcup_{i=1}^p M_i$ with $M_i \subset \operatorname{supp}(\mathcal{O}_n \cdot \mu_i)$. Then set

$$\nabla(I) = \{ a \in \mathcal{O}_n^p, \operatorname{supp}(a_i \cdot \mu_i) \subset M_i \text{ for all } i \}.$$

Any weight vector λ as above induces on $\mathcal{O}_n$ a Banach scale structure

$$| b |_s = \Sigma_\alpha | b_\alpha | s^{\lambda' \alpha} \quad \text{for } b = \Sigma_\alpha b_\alpha x^\alpha \in \mathcal{O}_n.$$

In the sequel when dealing with $\mathcal{O}_n$-linear maps $l : \mathcal{O}_n^p \to \mathcal{O}_n^q$ we consider on the domain space $\mathcal{O}_n^p$ the Banach scale structure defined by

$$| b |_s = \Sigma_i | b_i |_s$$

and on the target space $\mathcal{O}_n^q$ the one given by

$$| b |_s = \Sigma_i s^{\lambda_0(i-1)} | b_i |_s.$$

Division Theorem **5.1.** — *Let* $l : \mathcal{O}_n^p \to \mathcal{O}_n^q$ *be an* $\mathcal{O}_n$*-linear map, say* $l(a) = a.m = \Sigma_i a_i m_i$. *Assume that the* m_i*'s form a standard basis of* $\mathrm{I} = \mathrm{Im}\, l$. *Set* $\mathrm{K} = \mathrm{Ker}\, l$ *and let* $\Delta(\mathrm{I})$ *and* $\nabla(\mathrm{I})$ *be defined as above.*

a) $\mathrm{I} \oplus \Delta(\mathrm{I}) = \mathcal{O}_n^q$, $\mathrm{K} \oplus \nabla(\mathrm{I}) = \mathcal{O}_n^p$, *and these decompositions are compatible.*

b) There is a constant $c > 0$ *such that for all small* s *the following holds: For any* $e \in (\mathcal{O}_n^q)_s$ *the unique elements* $a \in \nabla(\mathrm{I})_s$ *and* $b \in \Delta(\mathrm{I})_s$ *with* $e = \Sigma a_i m_i + b$ *satisfy*

$$\begin{aligned}(\min_i | m_i |_s) . | a |_s + | b |_s &\leqslant \Sigma | a_i |_s | m_i |_s + | b |_s \\ &\leqslant c \,| \Sigma a_i m_i + b |_s = c \,| e |_s.\end{aligned}$$

Proof. — We may assume that all m_i are $\neq 0$. For $s > 0$ with $m_1, \ldots, m_p \in (\mathcal{O}_n^q)_s$ the continuous linear map

$$u_s : \nabla(\mathrm{I})_s \oplus \Delta(\mathrm{I})_s \to (\mathcal{O}_n^q)_s : (a, b) \mapsto a.m + b$$

will be shown to be bijective for small s. To this end supply the Banach space $\nabla(\mathrm{I})_s \oplus \Delta(\mathrm{I})_s$ with the norm

$$\| (a, b) \|_s = \Sigma_i | a_i |_s | \mu_i |_s + | b |_s.$$

By definition of $\nabla(\mathrm{I})$ the map

$$v_s : \nabla(\mathrm{I})_s \oplus \Delta(\mathrm{I})_s \to (\mathcal{O}_n^q)_s : (a, b) \mapsto a.\mu + b$$

is bijective, bicontinuous of norm 1, and its inverse v_s^{-1} has norm 1 as well. Decompose u_s into $u_s = v_s + w_s$ where $w_s(a, b) = a.m'$ with $m'_i = m_i - \mu_i$. There are constants $c > 0$ and $\varepsilon > 0$ such that

$$| \mu_i |_s \leqslant | m_i |_s \leqslant c \,| \mu_i |_s \quad \text{and} \quad | m'_i |_s \leqslant s^{\varepsilon} | \mu_i |_s$$

for all i and all sufficiently small s. This yields

$$| w_s | \leqslant s^{\varepsilon} \quad \text{and} \quad | w_s v_s^{-1} | \leqslant s^{\varepsilon} < 1$$

for small s, say $s \leqslant s_0$. Using the geometric series one sees that $u_s v_s^{-1} = \mathrm{id} + w_s v_s^{-1}$ is invertible with

$$| (u_s v_s^{-1})^{-1} | \leqslant \frac{1}{1 - s_0^{\varepsilon}} =: c'.$$

Consequently u_s is invertible and

$$| u_s^{-1} | \leqslant c' \quad \text{for } s \leqslant s_0.$$

This implies part *b)* of the Theorem. Since trivially $\mathrm{I}_s \cap \Delta(\mathrm{I})_s = 0$ we obtain moreover $\mathrm{I}_s \oplus \Delta(\mathrm{I})_s = (\mathcal{O}_n^q)_s$ and $\mathrm{K}_s \oplus \nabla(\mathrm{I})_s = (\mathcal{O}_n^p)_s$. This gives *a)*.

Example. — Consider $l : \mathcal{O}_2^2 \to \mathcal{O}_2$ given by $m_1 = x + y$, $m_2 = x - y$. Assume $\lambda_1 < \lambda_2$. Then $1 . m_1 - 1 . m_2 = 2y$, but there is no constant $c > 0$ such that for all small s

$$| 1 |_s \, | m_1 |_s + | -1 |_s \, | m_2 |_s = 2(s^{\lambda_1} + s^{\lambda_2})$$

is bounded by

$$c \, | 2y |_s = 2c . s^{\lambda_2}.$$

The reason is that m_1, m_2 are not a standard basis.

We now use the Division Theorem to estimate the norms of projections onto submodules $\mathrm{I} \subset \mathcal{O}_n^q$ and of scissions of $\mathcal{O}_n$-linear maps $l : \mathcal{O}_n^p \to \mathcal{O}_n^q$. Consider the Banach scale structures on $\mathcal{O}_n^p$ and $\mathcal{O}_n^q$ as defined above. Let $\mu_i = (0, \ldots, 0, x^{\alpha_i}, 0, \ldots, 0)$ with entry in the j_i-th place the unique minimal monomial generator system of $\mathrm{in}(\mathrm{I})$. The number

$$\beta(\mathrm{I}) = \max_i \lambda(j_i \, \alpha_i)$$

is called the *weighted order* of I w.r.t. the weight λ. It equals the maximal weighted order of the elements of a minimal standard basis of I.

Theorem **5.2.** — *Let $l : \mathcal{O}_n^p \to \mathcal{O}_n^q$ be an $\mathcal{O}_n$-linear map with image $\mathrm{I} = \mathrm{Im}\, l$ and $\beta = \beta(\mathrm{I})$.*

a) The projection $\pi_{\mathrm{I}} : \mathcal{O}_n^q \to \mathrm{I}$ induced by the decomposition $\mathcal{O}_n^q = \mathrm{I} \oplus \Delta(\mathrm{I})$ is compatible and satisfies for some constant $c > 0$ and all small $s > 0$

$$| (\pi_{\mathrm{I}})_s | \leqslant c.$$

b) l admits a compatible continuous linear scission $\sigma : \mathcal{O}_n^q \to \mathcal{O}_n^p$ satisfying for some constant $c > 0$ and all small $s > 0$

$$| \sigma_s | \leqslant c . s^{-\beta}.$$

Proof. — By the choice of the norms, *a)* follows from part *b)* of the Division Theorem. For *b)* choose a minimal standard basis $m_1, \ldots, m_{p'} \in \mathrm{I}$ with $\mu_i = \mathrm{in}\, m_i$. Consider the $\mathcal{O}_n$-linear map $l' : \mathcal{O}_n^{p'} \to \mathcal{O}_n^q : a \mapsto \Sigma \, a_i \, m_i$. The Division Theorem gives a compatible decomposition

$$\mathrm{Ker}\, l' \oplus \nabla'(\mathrm{I}) = \mathcal{O}_n^{p'}.$$

Then $\tau' = (l' |_{\nabla'})^{-1} : \mathrm{I} \to \mathcal{O}_n^{p'}$ is a compatible continuous linear section of l'. Moreover there is a constant $c' > 0$ such that

$$\Sigma \, | a_i |_s \, | \mu_i |_s \leqslant c' \, | \Sigma \, a_i \, m_i |_s$$

for $a \in \nabla'(\mathrm{I})$ and all small s. By definition of β we have $s^{\beta} = \min_i | \mu_i |_s$ and thus

$$| \tau'_s | \leqslant c' \, s^{-\beta}.$$

Choose an $\mathcal{O}_n$-linear map $\rho : \mathcal{O}_n^{p'} \to \mathcal{O}_n^p$ with commuting diagram

$$\begin{tikzcd} \mathcal{O}_n^{p'} \arrow[d, "\rho"'] \arrow[dr, "l'"] & \\ \mathcal{O}_n^p \arrow[r, "l"] & \mathcal{O}_n^q \end{tikzcd}$$

Observe that ρ is compatible with $| \rho_s | \leqslant c''$ for some constant $c'' > 0$ and small s. Then $\sigma = \rho\tau' \pi_{\mathrm{I}}$ is a compatible continuous linear scission of l satisfying a norm estimate as claimed.

Remarks. — *a)* For an $\mathcal{O}_n$-linear map $l : \mathcal{O}_n^p \to \mathcal{O}_n^q$ with image I one trivially has $\mathrm{Im}\, l_s \subset \mathrm{I}_s$. The existence of the compatible scission σ implies that $\mathrm{Im}\, l_s = \mathrm{I}_s$ for small s. This is false if instead of the Banach scale structure defined by **Z**-linearly independent weights $\lambda_0, \lambda_1, \ldots, \lambda_n$ as above we would have taken different Banach scale structures. For example define $| a |_s = \Sigma \, | a_{ij} | \, s^{i+j}$ for $a = \Sigma \, a_{ij} \, x^i y^j \in \mathbf{C}\{ x, y \} = \mathcal{O}_2$. Let $l : \mathcal{O}_2 \to \mathcal{O}_2 : a \mapsto a.(x-y)$ with $\mathrm{I} = \mathrm{Im}\, l = (x-y)$. For fixed $s > 0$ set

$$a = \sum_{n=0}^{\infty} \frac{1}{k^2 \, s^k} \sum_{i=0}^{k-1} x^i y^{k-1-i}.$$

We then have

$$| a.(x-y) |_s = \sum_{k=0}^{\infty} \frac{2}{k^2} < \infty$$

$$| a |_s = \sum_{k=0}^{\infty} \frac{1}{k.s} = \infty.$$

Thus $a.(x-y) \in \mathrm{I}_s$ but $\notin \mathrm{Im}\, l_s$.

b) We can now prove the corollary of section 1: Let $f : \mathrm{U} \to \mathcal{O}_n^q$ be an analytic map in a neighborhood $\mathrm{U} \subset \mathcal{O}_n^p$ of 0 given by substitution in some $g(x, y)$. Then f has constant rank at 0 if and only if f is flat at 0. In this case f is of constant rank at 0 w.r.t. $\Delta(\mathrm{I})$, where $\mathrm{I} = \mathrm{Im}\, l$. Indeed, let σ be the scission constructed above. Then $\mathrm{Ker}\, l\sigma = \mathrm{Ker}\, \pi_{\mathrm{I}} = \Delta(\mathrm{I})$. Now our claim follows from the results of section 1.

c) For any submodule I of $\mathcal{O}_n^q$ the weighted order $\beta(m_n^c \, \mathrm{I})$ increases at most linearly with coefficient 1 in c. To see this, consider the inclusions

$$m_n^c . \mathrm{in}(\mathrm{I}) \subset \mathrm{in}(m_n^c \, \mathrm{I}) \subset \mathrm{in}(\mathrm{I}).$$

Let μ be an element of the minimal monomial generator system of $\mathrm{in}(m_n^c \, \mathrm{I})$. Then $\mu = x^{\alpha} \nu$ with ν an element of the minimal monomial generator system of $\mathrm{in}(\mathrm{I})$ and

$x^\alpha \in \mathcal{O}_n$. If $|\alpha| > c$, μ belongs to $m_n^c.\mathrm{in}(I)$ but is not a minimal monomial vector of this module w.r.t. the componentwise order on $\mathbf{N}^n$. The first inclusion implies that it neither is minimal in $\mathrm{in}(m_n^c\, I)$ which is a contradiction. Therefore $|\alpha| \leqslant c$ and $\beta(m_n^c\, I) \leqslant \beta(I) + c$.

6. Banach scales

For details and the basic notions of this and the next section we refer the reader to [BS, He, Gr, U]. A *Banach scale* is a topological vector space E, Hausdorff, locally convex, and sequentially complete together with pseudonorms $|\ |_s$, $s \in \mathbf{R}$ and $s > 0$, such that

(i) if $s < s'$ then $|\ |_s \leqslant |\ |_{s'}$,
(ii) $E_s := \{ a \in E, |a|_s < \infty \}$ is a Banach space with norm $|\ |_s$,
(iii) $E = \bigcup_s E_s$ as topological vector space with the final topology.

We often say that E is a Banach scale. But the reader is reminded that the Banach spaces E_s with their fixed norms $|\ |_s$ are part of the structure.

Remarks. — *a)* For $r \in \mathbf{R}$, $r > 0$, let $B_s(0, r) = \{ a \in E_s, |a|_s < r \}$ be the open ball of radius r. Then $B(0, r) := \bigcup_s B_s(0, r)$ is open in E. Indeed, for $s < s'$ and $a \in E_{s'} \subset E_s$ we have $|a|_s \leqslant |a|_{s'}$, hence $B_{s'}(0, r) \subset B_s(0, r)$. Therefore

$$B(0, r) \cap E_{s'} = B_{s'}(0, r) \cup \bigcup_{s < s'} (B_s(0, r) \cap E_{s'})$$

is open in $E_{s'}$ as $E_{s'} \subset E_s$ is continuous for $s < s'$.

b) Let F be a closed linear subspace of a Banach scale $E = \bigcup_s E_s$. Then $F = \bigcup_s F_s$ with $F_s = E_s \cap F$ is a Banach scale. In fact, it is immediately seen that a subset $A \subset F$ is relatively closed if and only if $A \cap F_s$ is closed in F_s for all s. Thus the final topology of F with respect to the inclusions $F_s \subset F$ coincides with the relative topology, which proves (iii).

c) Let F be a Banach scale. Every decomposition $F = I \oplus J$ as an algebraic direct sum of closed linear subspaces is in fact topological. To show this it is enough to prove that the projection $\pi : F \to I$ with kernel J is continuous. By the Closed Graph Theorem [Gr, chapter 4.1.5, Theorem 2] it suffices to show that graph π is closed. But this is obvious from

$$\mathrm{graph}\ \pi = \{ (a, b) \in F \oplus I,\ a - b \in J \}.$$

A continuous linear map $l : E \to F$ between Banach scales is called *compatible* if, for small s, l restricts to a map $l_s : E_s \to F_s$. Then l_s is necessarily a continuous linear map. Since the relative topology on F_s induced from F is a Hausdorff topology coarser than its Banach space topology this follows from the Closed Graph Theorem, see [Gr, chapter 1.14, Corollary to Theorem 10].

A decomposition $F = I \oplus J$ as a direct sum of closed linear subspaces is called *compatible* if $F_s = I_s \oplus J_s$ with $I_s = F_s \cap I$, $J_s = F_s \cap J$ for small s. Then clearly the projections onto I and J are compatible.

An analytic (i.e. continuous and Gâteaux-analytic) map $f: \mathrm{U} \to \mathrm{F}$, $\mathrm{U} \subset \mathrm{E}$ open is called *compatible* at $a \in \mathrm{U}$ if, for small s, f restricts to an analytic map $f_s: \mathrm{U}_s \to \mathrm{F}_s$ where U_s is a suitable neighborhood of a contained in $\mathrm{U} \cap \mathrm{E}_s$. In that case the chain rule gives that $T_a f_s = T_a f|_s$ for the Gâteaux-differentials $T_a f: \mathrm{E} \to \mathrm{F}$ and $T_a f_s: \mathrm{E}_s \to \mathrm{F}_s$.

Proposition **6.1.** — *Let* E *and* F *be Banach scales,* $\mathrm{U} \subset \mathrm{E}$ *an open set and* $f: \mathrm{U} \to \mathrm{F}$ *a map. Then f is analytic if and only if, for all analytic curves* $\gamma: \mathrm{T} \to \mathrm{E}$ *with* $\mathrm{T} \subset \mathbf{C}$ *open and* $\gamma(\mathrm{T}) \subset \mathrm{U}$, *the composition* $f \circ \gamma: \mathrm{T} \to \mathrm{F}$ *is analytic.*

Proof. — [He, Theorem 3.2.7 *d)*].

Inverse Mapping Theorem **6.2.** — *Let* $\mathrm{E} = \bigcup_s \mathrm{E}_s$, $\mathrm{F} = \bigcup_s \mathrm{F}_s$ *be Banach scales, and let* $f: \mathrm{U} \to \mathrm{F}$ *be an analytic map in a neighborhood* $\mathrm{U} \subset \mathrm{E}$ *of* 0 *with* $f(0) = 0$. *Assume:*

(i) *The Gâteaux-differential* $l = T_0 f: \mathrm{E} \to \mathrm{F}$ *is an isomorphism of topological vector spaces.*
(ii) *The composition* $l^{-1} \circ f$ *is compatible at* 0 *with restrictions* $(l^{-1} \circ f)_s: \mathrm{U}_s \to \mathrm{F}_s$.
(iii) *There are constants* $0 < c < 1/e(e+1)$ *and* $r > 0$ *such that for all small* $s > 0$

$$|(l^{-1} \circ f)_s - \mathrm{id}_{\mathrm{E}_s}|_r \leqslant c.r.$$

Then f is a local analytic isomorphism at 0.

Proof. — We proceed exactly as in the usual proof of the Inverse Mapping Theorem in Banach spaces. Upon replacing f by $l^{-1} \circ f$ we may assume $\mathrm{F} = \mathrm{E}$ and $l = T_0 f = \mathrm{id}_{\mathrm{E}}$. Write $f = \mathrm{id}_{\mathrm{E}} + h$ and define recursively

$$g^0 = 0, \qquad g^{n+1} = \mathrm{id}_{\mathrm{E}} - h \circ g^n.$$

As $\operatorname{ord} h \geqslant 2$ it is shown by induction (using Proposition 6.3 below) that $\operatorname{ord}(g^{n+1} - g^n) > n$. Let $\bar{g}$ be the unique formal power series satisfying $\operatorname{ord}(\bar{g} - g^n) > n$ for all n. Then also $\operatorname{ord}(f \circ \bar{g} - f \circ g^n) > n$. Since $\operatorname{ord}(f \circ g^n - \mathrm{id}_{\mathrm{E}}) = \operatorname{ord}(g^n - g^{n+1}) > n$ we conclude that $f \circ \bar{g} = \mathrm{id}_{\mathrm{E}}$, i.e. $\bar{g}$ is a formal right inverse of f. We shall show that $\bar{g}$ converges on the open set $\mathrm{B}(0, r') = \bigcup_s \mathrm{B}_s(0, r')$ for $r' = ((1/e) - c).r$. Note that $r' > 0$ since $c < 1/e$. Let us first prove by induction that

$$|g_s^n|_{r'} \leqslant \frac{r}{e}$$

for small s. Indeed, using assumption (iii) we have

$$\begin{aligned} |g_s^n|_{r'} &= |(\mathrm{id}_{\mathrm{E}} - h \circ g^{n-1})_s|_{r'} \\ &\leqslant r' + |h_s|_{e.|g_s^{n-1}|_{r'}} \\ &\leqslant r' + |h_s|_r \\ &\leqslant r' + c.r = \frac{r}{e}. \end{aligned}$$

But then, since $\operatorname{ord}(\bar{g} - g^n) > n$, we also have

$$| \bar{g}_s |_{r'} \leqslant \frac{r}{e}.$$

This implies that $\bar{g}$ induces analytic maps $g_s : \mathrm{B}_s(0, r') \to \mathrm{E}_s$ with $| g_s |_{r'} \leqslant r/e$ as well as an analytic map $g : \mathrm{B}(0, r') \to \mathrm{E}$. Thus the following holds:

$$f \circ g = \mathrm{id}_{\mathrm{B}(0,r')}, \quad g(0) = 0, \quad T_0 g = \mathrm{id}_{\mathrm{E}}$$

$$| g_s - \mathrm{id}_{\mathrm{E}_s} |_{r'} \leqslant c'.r' \quad \text{with } c' = \frac{ce}{1 - ce}.$$

Using $c < 1/e(e+1)$ we see that $c' < 1/e$. Hence the whole argument above can be applied to g with r and c replaced by r' and c'. Setting $r'' = ((1/e) - c').r'$ we find an analytic map $\tilde{f} : \mathrm{B}(0, r'') \to \mathrm{E}$ with $g \circ \tilde{f} = \mathrm{id}_{\mathrm{B}(0,r'')}$. Uniqueness of the inverse implies $\tilde{f} = f\,|_{\mathrm{B}(0,r'')}$. Thus $f : \mathrm{B}(0, r'') \to f(\mathrm{B}(0, r''))$ is an analytic isomorphism.

Example. — We indicate what can happen if the norm estimate (iii) in the Inverse Mapping Theorem is violated. View $\mathrm{E} = \mathcal{O}_1 = \mathbf{C}\{x\}$ as a Banach scale as in section 5. Then $\pi : \mathrm{E} \to \mathrm{E} : a \mapsto a - a(0)$ is a compatible continuous linear map. Define $f : \mathrm{E} \to \mathrm{E}$ by

$$f(a) = a + \frac{a^2 - a^2(0)}{x} = a + \frac{\pi(a^2)}{x}.$$

It is analytic and compatible at 0. Its Gâteaux-differential is given by

$$T_a f(b) = b + \frac{\pi(2ab)}{x}.$$

In particular, $T_0 f = \mathrm{id}_{\mathrm{E}}$. Nevertheless f is not a local analytic isomorphism at 0. In fact, if $a \in \mathrm{E}$ is invertible, $a(0) \neq 0$, we have for $b := a(0).(x + 2a)^{-1}$ that $T_a f(b) = (1/x)\, \pi((x + 2a)\, b) = 0$. Hence there are points $a \in \mathrm{E}$ arbitrarily close to 0 such that $T_a f$ is not injective. This phenomenon can be explained by comparing the size of the terms of order $\geqslant 2$ at 0 with $T_0 f = \mathrm{id}_{\mathrm{E}}$. For this let us calculate $| f_s - \mathrm{id}_{\mathrm{E}_s} |_r$. We have $f(a) = a + \mathrm{P}(a)$ where $\mathrm{P}(a) = (1/x).\pi(a^2)$ is a continuous homogeneous polynomial of degree 2. For fixed s we have $| \pi(a^2) |_s = | a^2 - a^2(0) |_s \leqslant | a^2 |_s \leqslant | a |_s^2$. If $a = x.s^{-1}$ then $| a |_s = 1$ and $| \pi(a^2) |_s = 1$. This implies $| \mathrm{P}_s | = s^{-1}$ and

$$| f_s - \mathrm{id}_{\mathrm{E}_s} |_r = | \mathrm{P}_s |_r = r^2.s^{-1}.$$

But we cannot find a constant c such that $r^2.s^{-1} \leqslant c.r$ for all small s. Note that $T_a f$ is injective for $a \in (x)$, say of order $\geqslant 1$. Moreover, $T_a f : \mathcal{O}_1 \to \mathcal{O}_1$ is surjective for all a close to 0. Indeed, $x + 2a$ is not in $(x)^2$ for a close to 0. Hence given $c \in \mathcal{O}_1$ we can solve $cx = (x + 2a)\, b$ for b. Then $T_a f(b) = c$.

Proposition **6.3.** — *Let* $f: \mathrm{U} \to \mathrm{F}$, $g: \mathrm{V} \to \mathrm{G}$ *be analytic with* $\mathrm{U} \subset \mathrm{E}$, $\mathrm{V} \subset \mathrm{F}$ *open and* $f(\mathrm{U}) \subset \mathrm{V}$. *Then* $g \circ f: \mathrm{U} \to \mathrm{G}$ *is analytic and for* $a \in \mathrm{U}$

$$T_a(g \circ f) = T_{f(a)}\, g \circ T_a f.$$

Moreover, if $0 \in \mathrm{U}$, $f(0) = 0 \in \mathrm{V}$ *and* $g(0) = 0$, *the expansion of* $g \circ f$ *at* 0 *is given by the composition formula for formal power series applied to the expansions of* g *and* f *at* 0.

Proof. — The first part is standard. For the second, one can assume by the Hahn-Banach-Theorem that $\mathrm{E} = \mathrm{G} = \mathbf{C}$. Let $g = \Sigma\, g_i$ be the expansion of g in homogeneous polynomials and $\hat{g}_i$ be the corresponding i-linear symmetric maps. Let then i be fixed. As $\hat{g}_i$ is continuous there are a constant $c > 0$ and a continuous seminorm p on F such that

$$|\, \hat{g}_i(b_1, \ldots, b_i) \,| \leqslant c . p(b_1) \ldots p(b_i) \quad \text{for all } b_1, \ldots, b_i \in \mathrm{F}.$$

Fix a near 0. By [BS, Proposition 4.1] there are constants $\mathrm{M} > 0$ and $\theta \in \,]0, 1[$ such that

$$p(f_k(a)) \leqslant \mathrm{M} . \theta^k \quad \text{for all } k.$$

Consequently

$$\Sigma_{k \geqslant i}\, \Sigma_{k_1 + \ldots + k_i = k}\, |\, \hat{g}_i(f_{k_1}(a), \ldots, f_{k_i}(a)) \,| \leqslant \Sigma\, \Sigma\, c\mathrm{M}^i\, \theta^k = c\mathrm{M}^i\, \Sigma_{k \geqslant i} \binom{k-1}{i-1} \theta^k < \infty.$$

As $\hat{g}_i$ is i-linear and continuous we obtain

$$g_i(f(a)) = \hat{g}_i(f(a), \ldots, f(a)) = \Sigma_{k \geqslant i}\, \Sigma_{k_1 + \ldots + k_i = k}\, \hat{g}_i(f_{k_1}(a), \ldots, f_{k_i}(a)).$$

This is the power series expansion of $g_i \circ f$. For sufficiently small $r > 0$ we have

$$g(f(a)) = \Sigma_i\, g_i(f(a)) \quad \text{for } |\, a \,| < r,$$

the series converging uniformly on $|\, a \,| < r$, see [BS, proof of Theorem 6.4]. By Weierstrass' Theorem on (locally) uniform sequences of analytic functions

$$g(f(a)) = \Sigma_k\, \Sigma_{i \leqslant k}\, \Sigma_{k_1 + \ldots + k_i = k}\, \hat{g}_i(f_{k_1}(a), \ldots, f_{k_i}(a))$$

is the power series expansion on $|\, a \,| < r$.

7. Analytic maps between power series spaces

We now specialize to power series spaces. Equip $\mathcal{O}_n^v$ with pseudonorms defined through weight vectors as in section 5. Then $\mathcal{O}_n^v(s) = \{\, a \in \mathcal{O}_n^v,\ |\, a \,|_s < \infty \,\}$ is a Banach space and $\mathcal{O}_n^v = \bigcup_s \mathcal{O}_n^v(s)$ with the final topology a Banach scale [GR]. Partial differentiation and integration $\mathcal{O}_n \to \mathcal{O}_n$ as well as algebra homomorphisms $\mathcal{O}_n \to \mathcal{O}_m$ and $\mathcal{O}_n$-module homomorphisms are continuous linear maps, hence analytic. The next result characterizes analytic maps with finite-dimensional domain [Mü2, sec. 6]:

Proposition **7.1.** — *a) For a map* $f: \mathrm{T} \to \mathcal{O}_n^q$, $\mathrm{T} \subset \mathbf{C}^p$ *open, the following conditions are equivalent:*

(i) f *is analytic.*

(ii) f *is continuous and for all jet maps* $\pi_c : \mathcal{O}_n^q \to (\mathcal{O}_n/m_n^{c+1})^q$ *the composition* $\pi_c \circ f$ *is analytic.*

(iii) *For every* $t_0 \in \mathrm{T}$ *there are open neighborhoods* $\mathrm{T}' \subset \mathrm{T}$ *of* t_0, $\mathrm{V} \subset \mathbf{C}^n$ *of* 0 *and an analytic map* $g: \mathrm{V} \times \mathrm{T}' \to \mathbf{C}^q$ *such that for* $t \in \mathrm{T}' : f(t)\,(x) = g(x, t)$.

(iv) *For every* $t_0 \in \mathrm{T}$ *there are an open neighborhood* $\mathrm{T}' \subset \mathrm{T}$ *of* t_0 *and an* $s > 0$ *such that* $f(\mathrm{T}') \subset \mathcal{O}_n^q(s)$ *and* $f: \mathrm{T}' \to \mathcal{O}_n^q(s)$ *is analytic.*

b) Assume that (i) *to* (iv) *hold. Let* $g(x, t_0 + t) = \Sigma_{\alpha, \beta}\, g_{\alpha\beta}\, x^\beta t^\alpha$ *be the power series expansion of* g *near* $(0, t_0)$. *For* $t \in \mathbf{C}^p$ *let*

$$f_k(t) = \Sigma_{|\alpha| = k} \Sigma_\beta\, g_{\alpha\beta}\, x^\beta t^\alpha \in \mathcal{O}_n^q.$$

Then $f(t_0 + t) = \Sigma_k f_k(t)$ *is the power series expansion of* f *near* t_0.

c) For $t \in \mathrm{T}'$ *as in* (iii) *above, the Gâteaux-differential* $T_t f: \mathbf{C}^p \to \mathcal{O}_n^q$ *is given by*

$$T_t f(b_1, \ldots, b_p) = \Sigma_i\, \partial_{t_i} g(x, t) \,.\, b_i.$$

The main examples of analytic maps between power series spaces are given by substitution:

Proposition **7.2.** — *Let* $g: \mathrm{V} \times \mathrm{W} \to \mathbf{C}^q$ *be analytic, where* $\mathrm{V} \times \mathrm{W} \subset \mathbf{C}^n \times \mathbf{C}^p$ *is an open neighborhood of* 0. *Let* $\mathrm{U} = \{ a \in \mathcal{O}_n^p,\ a(0) \in \mathrm{W} \}$, *and define* $f: \mathrm{U} \to \mathcal{O}_n^q$ *by substitution in* $g: f(a)\,(x) = g(x, a)$.

a) f *is analytic. For* $a \in \mathrm{U}$ *its Gâteaux-differential* $T_a f: \mathcal{O}_n^p \to \mathcal{O}_n^q$ *is given by*

$$T_a f(b_1, \ldots, b_p) = \Sigma_i\, \partial_{y_i} g(x, a) \,.\, b_i.$$

b) Let $g(x, y) = \Sigma_{\alpha, \beta}\, g_{\alpha\beta}\, x^\beta y^\alpha$ *be the expansion of* g. *For* $a \in \mathcal{O}_n^p$ *let*

$$f_k(a) = \Sigma_{|\alpha| = k} \Sigma_\beta\, g_{\alpha\beta}\, x^\beta a^\alpha \in \mathcal{O}_n^q.$$

Then $f(a) = \Sigma_k f_k(a)$ *is the power series expansion of* f *near* 0.

c) f *is compatible at* 0.

Proof. — We may assume $q = 1$. *a)* Propositions 6.1 and 7.1 imply that f is analytic. For $\alpha \in \mathrm{U}$ and $b \in \mathcal{O}_n^p$, let $h(t) = f(a + tb)$, $t \in \mathbf{C}$ close to 0. Then

$$h(t)\,(x) = g(x, a + tb) =: \widetilde{h}(x, t).$$

By Proposition 7.1 *c)* we have

$$T_a f(b) = \frac{d}{dt} h(t)\,\big|_{t=0} = \partial_t \widetilde{h}(x, t)\,\big|_{t=0} = \Sigma_i\, \partial_{y_i} g(x, a) \,.\, b_i.$$

b) For small s the f_k are continuous homogeneous polynomials $\mathcal{O}_n^p(s) \to \mathcal{O}_n(s)$ of degree k. For $a \in \mathrm{B}_s(0, r)$, r sufficiently small, the series $\Sigma_k f_k(a)$ converges to $f(a)$ in $\mathcal{O}_n(s)$. Hence f restricts to an analytic map $\mathrm{B}_s(0, r) \to \mathcal{O}_n(s)$. This proves *b)* and *c)*.

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